Laplace Transform

Introduction

The Laplace transform provides a useful method of solving certain types of differential equations when certain initial conditions are given, especially when the initial values are zero. The Laplace transforms is also very useful in the area of circuit analysis. It is often easier to analyze the circuit in its Laplace form, than to form differential equations. The techniques of Laplace transform are not only used in circuit analysis, but also in:

- Proportional-Integral-Derivative (PID) controllers
- DC motor speed control systems
- DC motor position control systems
-Second order systems of differential equations (under damped, over damped and critically damped)

The Laplace transform of a function f(t) for t > 0 is defined by the following integral defined over 0 to ∞:

$$\int_0^\infty e^{-st} f(t)\, dt \qquad (1)$$

The resulting expression is a function of s, which we write as F(s). In words we say "The Laplace Transform of f(t) equals function F of s″ and write:

$$L\{f(t)\} = F(s) \qquad (2)$$

Table of Laplace Transformations

f(t)	L{f(t)} = F(s)	Examples
c	c/s	$L\{3\}= 3/s$
e^{at}	$1/(s-a)$	$L\{e^{-2t}\}=1/(s+2)$
cosat	$s/(s^2+ a^2)$	$L\{\cos 3t\}= s/(s^2+ 9)$
sinat	$a/(s^2+ a^2)$	$L\{\sin 2t\}= 2/(s^2+ 4)$
coshat	$s/(s^2- a^2)$	$L\{\cosh(5)t\}= s/(s^2- 25)$
sinhat	$a/(s^2- a^2)$	$L\{\sinh 4t\}= 4/(s^2- 16)$
t^n	$\frac{\Gamma n+1}{s^{n+1}}, \Gamma n+1=n!$	$L\{t^4\} = \frac{\Gamma 5}{s^5}$
e^{at} coskt	$\frac{s-a}{(s-a)^2+k^2}$	$L\{e^{-4t}\cos 3t\}= \frac{s+4}{(s+4)^2+9}$
e^{at} sinkt	$\frac{k}{(s-a)^2+k^2}$	$L\{e^{-t}\sin\sqrt{3}\,t\}=\frac{\sqrt{3}}{(s+1)^2+3}$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(F(s))$	$L\{t^2\sin 2t\}=(-1)^2\frac{d^2}{ds^2}[\frac{2}{s^2+4}]$ $=\frac{-3s^4-8s^2+16}{(s^2+4)^4}$
$\int_{u=0}^{t} f(u)\,du$	$\frac{F(s)}{s}$	$L\{\int_{u=0}^{t}\cos(5u)\,du\} = \frac{1}{s^2+25}$

$\frac{f(t)}{t}$	$\int_{s=0}^{\infty} F(s)\,ds$	$L\left\{\frac{\cos 2t-\cos 3t}{t}\right\}$ $=\int_{s=0}^{\infty}\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)ds$ $=\frac{1}{2}\ln\left(\frac{s^2+9}{s^2+4}\right)$

Some properties of Laplace Transforms

1) If a and b are constants while f(t) and g(t) are functions of t, then

$$L\{a\,f(t) + b\,g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$$

2) If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$

3) $L\{\,e^{at}\,f(t)\} = F(s-a)$

4) $L\{\,t\,f(t)\} = -\,F`(s)$

By using the definition such that $L\{f(t)\} = F(s)$, we have to get Laplace of the above functions such that:

1- $L\{c\} = \int_0^{\infty} c\,e^{-st}dt = -[c\,e^{-st}/s]_{t=0}^{\infty} = c/s$

2-$L\{e^{at}\} = \int_0^\infty e^{at}\, e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = -[e^{-(s-a)t}/(s-a)]_{t=0}^{\infty} = 1/(s-a)$

By using $L\{e^{at}\} = 1/(s-a)$, we can get $L\{\cosh at\}$ and $L\{\sinh at\}$ so that:

$L\{\cosh at\} = L\{ (e^{at} + e^{-at})/2\} = (½)[L\{e^{at}\} + L\{e^{-at}\}]$

$= (1/2)[1/(s-a) + 1/(s+a)] = s/(s^2 - a^2)$

$L\{\sinh at\} = L\{(e^{at} - e^{-at})/2\} = (½)[L\{e^{at}\} - L\{e^{-at}\}]$

$= (1/2)[1/(s-a) - 1/(s+a)] = a /(s^2 - a^2$

3- $L\{\cos at\}$ and $L\{\sin at\}$ can be evaluated using integration by parts such that:

$$L\{\cos at\} = \int_0^\infty \cos at\, e^{-st} dt = [e^{-st}(\sin at)/a]_{t=0}^{\infty} + (s/a)\int_0^\infty \sin at\, e^{-st} dt$$

$$= (s/a)([-e^{-st}(\cos at)/a]_{t=0}^{\infty} - (s/a)\int_0^\infty \cos at\, e^{-st} dt)$$

Therefore $(s^2 + a^2)/(a^2) \int_0^\infty \cos at\, e^{-st} dt = s/a^2$, thus

$$L\{\cos at\} = \int_0^\infty \cos at\, e^{-st} dt = s/(s^2 + a^2)$$

Also $L\{\sin at\}$

$$= \int_0^\infty \sin at\, e^{-st} dt = [-e^{-st}(\cos at)/a]_{t=0}^{\infty} - (s/a)\int_0^\infty \cos at\, e^{-st} dt$$

$= [1/a] - (s/a)\ ([e^{-st}(\sin at)/a]_{t=0}^{\infty} + (s/a)\int_0^{\infty} \sin at\, e^{-st} dt)$

Therefore $(s^2 + a^2)/(a^2) \int_0^{\infty} \sin at\, e^{-st} dt = 1/a$, thus

$$L\{\sin at\} = \int_0^{\infty} \sin at\, e^{-st} dt = a/(s^2 + a^2)$$

4- To evaluate $L\{t\ f(t)\}$ using definition, we have to differentiate $F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ with respect to s, such that

$\frac{dF}{ds} = \int_0^{\infty} \frac{\partial}{\partial s} f(t)\, e^{-st}\, dt$, therefore $F`(s) = \int_0^{\infty} -t\, f(t)\, e^{-st}\, dt$, thus

$L\{t\ f(t)\} = \int_0^{\infty} t\, f(t)\, e^{-st}\, dt = -F`(s)$.

If we differentiate $\int_0^{\infty} -t\, f(t)\, e^{-st} dt$ with respect to s, therefore

$F``(s) = \int_0^{\infty} t^2 f(t)\, e^{-st}\, dt$, therefore $L\{t^2\ f(t)\} = F``(s)$ and by induction, we will get $L\{t^n\ f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$.

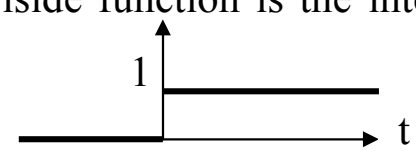
5- $L\{e^{at} f(t)\} = \int_0^{\infty} e^{at} f(t)\, e^{-st}\, dt = \int_0^{\infty} f(t)\, e^{-(s-a)t}\, dt = F(s-a)$

6-Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, thus $\int_s^\infty F(s)ds = \int_s^\infty (\int_0^\infty e^{-st} f(t)\,dt)\,ds$

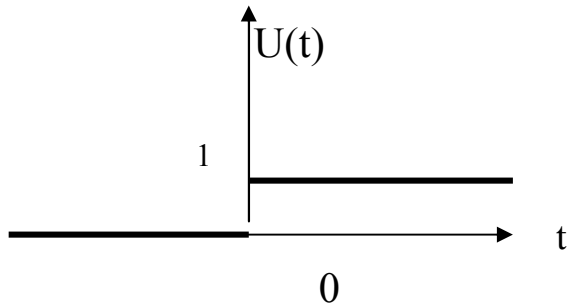
$$= \int_0^\infty (\int_s^\infty e^{-st} f(t)\,ds)dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = L\{\frac{f(t)}{t}\} = \int_s^\infty F(s)ds.$$

Unit Step function

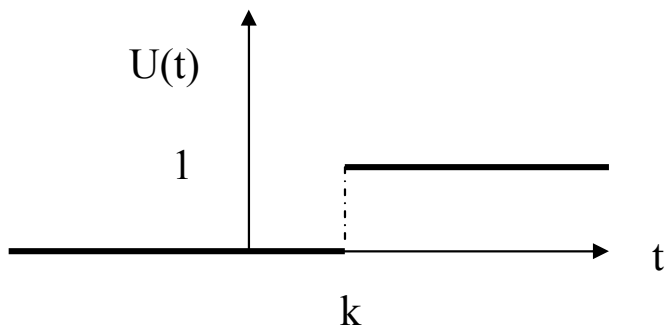
The Heaviside step function, H, also called the unit step function, is a discontinuous function whose value is zero for negative argument and one for positive argument. It seldom matters what value is used for H(0), since H is mostly used as a distribution. Some common choices can be seen below. The function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely. It was named after the English polymath Oliver Heaviside. It is the cumulative distribution function of a random variable which is almost surely 0. The Heaviside function is the integral of the Dirac delta function.



An alternative form of the unit step, as a function of a discrete variable n such that $U(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$



Unit step function is defined in general by $U(t-k) = \begin{cases} 0 & t < k \\ 1 & t > k \end{cases}$



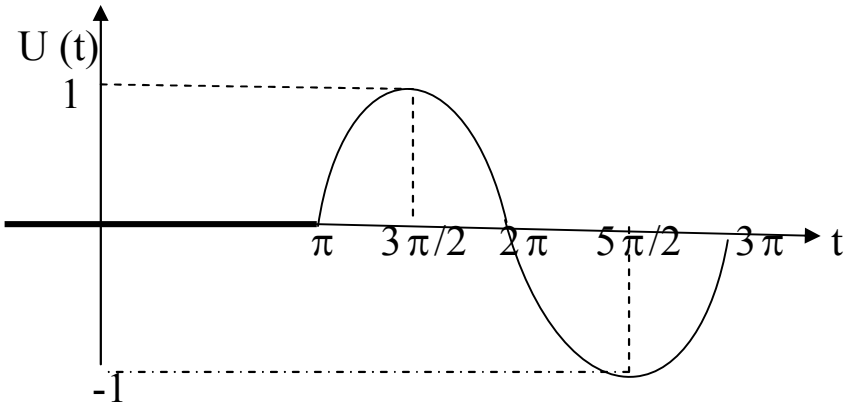
U(t-k) is denoted by notation $H_k(t)$ and if it is multiplied by a function f(t) , then $f(t-k)\, U_k(t) = \begin{cases} 0 & t < k \\ f(t) & t > k \end{cases}$

Example 1: Sketch the following functions

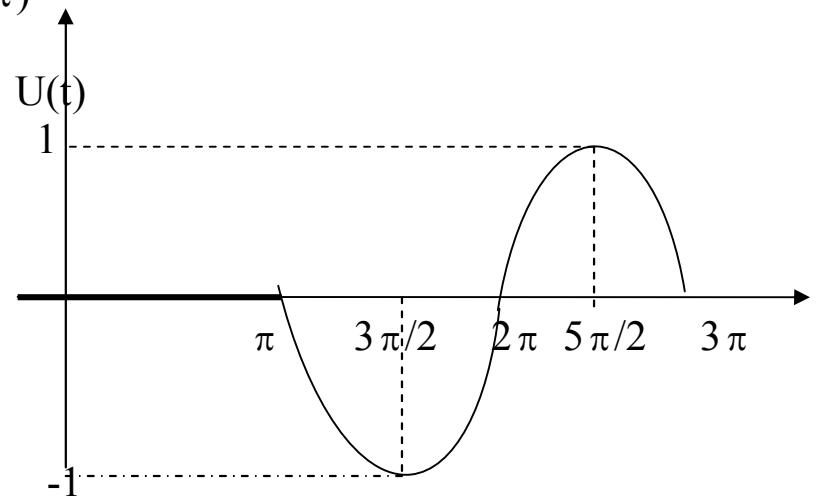
[sin (t-π) U (t-π) and sint U (t-π)]

Solution

sin (t-π) U (t-π)



sin(t) U (t-π)



Laplace transform of unit step function

The following properties allow us to get Laplace transform of unit step function such that:

a- $L\{U(t)\} = 1/s$

b- $L\{ f(t)\, U(t)\} = F(s)$

c- $L\{U(t-a)\} = e^{-as}/s$

d- $L\{f(t-a)\, U(t-a)\} = F(s)\, e^{-as}$

Proof

a- Since $U(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, thus

$$L\{U(t)\} = \int_0^\infty e^{-st}\, U(t)\, dt = \int_0^\infty e^{-st}\, dt = [e^{-st}/-s]_0^\infty = 1/s$$

b- And $L\{ f(t)\, U(t)\} = \int_0^\infty e^{-st}\, f(t)\, U(t)\, dt = \int_0^\infty e^{-st} f(t) dt = F(s)$

c- Since $U(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$, thus

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\, U(t-a)\, dt = \int_a^\infty e^{-st}\, dt = [e^{-st}/-s]_a^\infty = (1/s)e^{-as}$$

d- $L\{f(t-a)\,U(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,U(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)dt$

$= \int_0^\infty e^{-s(u+a)} f(u)du = e^{-as}\int_0^\infty e^{-su} f(u)du = e^{-as}\,F(s)$

Example 2: Find Laplace Transform for the following functions

i) $f(t) = t^3 + e^{-2t}\cos 5t$, ii) $f(t) = \begin{cases} 3 & 0 \le t \le 2 \\ 2t-1 & 2 < t \le 5 \\ 9 & t > 5 \end{cases}$

Solution

i) $F(s) = \dfrac{\Gamma 4}{4!} + \dfrac{s+2}{(s+2)^2+25}$

ii) Using unit step function so that

$f(t) = 3[\,U(t) - U(t-2)] + (2t-1)[\,U(t-2) - U(t-5)] + 9\,U(t-5)$

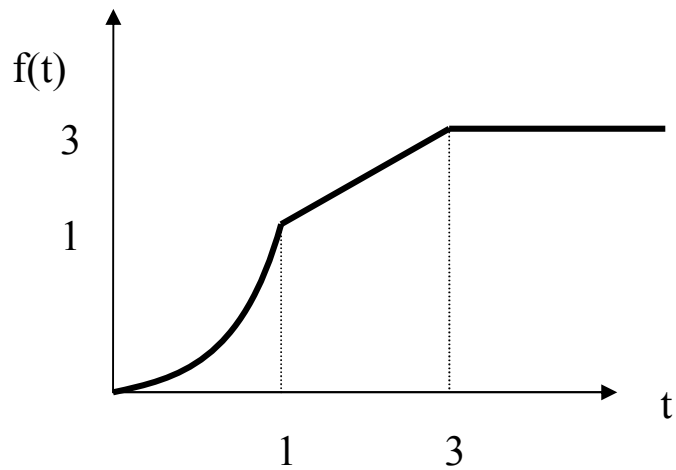
$= 3[\,U(t) - U(t-2)] + [2(t-2)+3]\,U(t-2) - [2(t-5)+9]\,U(t-5) + 9\,U(t-5) = 3\,U(t) + 2(t-2)U(t-2) - 2(t-5)U(t-5)$, therefore

$$F(s) = 3/s + (2/s^2)\,e^{-2s} - (2/s^2)\,e^{-5s}$$

Example 3: Sketch and find Laplace Transform for the function

$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ t & 1 < t < 3 \\ 3 & t > 3 \end{cases}$$

Solution:



$f(t) = t^2(U(t) - U(t-1)) + t(U(t-1)- U(t-3)) + 3 U(t-3) = t^2 U(t) - (t-1)^2 U(t-1) + (-2t+1) U(t-1) + (t-1)U(t-1) + U(t-1) - (t-3) U(t-3) - 3 U(t-3) + 3U(t-3) = t^2 U(t) - (t-1)^2 U(t-1) - 2(t-1) U(t-1) - (t-3) U(t-3)$, therefore

$$L\{f(t)\} = \frac{2}{s^3} - \frac{2}{s^3}e^{-s} - \frac{2e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}$$

Laplace transform of periodic function

If function f(t) is Periodic with period p > 0, so that f(t + p) = f(t), and $f_1(t)$ is one period (i.e. one cycle) of the function, written using Unit Step functions, then $L\{f(t)\} = \dfrac{L\{f_1(t)\}}{1-e^{-sp}}$,

where $L\{f_1(t)\} = \int_0^p f(t)\, e^{-st} dt$.

Example 4: Find Laplace transform of

$$f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 \le t < 4 \end{cases}, \text{ where } f(t+4) = f(t).$$

Solution

$$L\{f(t)\} = \left[\frac{\int_0^2 3t\, e^{-st} dt + \int_2^4 6\, e^{-st} dt}{1-e^{-4s}} = \frac{3(1-e^{-2s} - 2s\, e^{-4s})}{(1-e^{-4s})s^2}\right]$$

Problems on Laplace transform

Find F(s) of the following functions:

a) $f(t) = 5t+4\sin 2t-2\cosh 3t$, b) $f(t) = te^{-2t}- 4\sin^2 3t$

c) $f(t) = e^{-4t}- 5t^4 +\sin 3t\cosh 2t$, d) $f(t) = t\sin 2t\cos 3t$

e) $f(t) = \sin t\, U(t-\pi) + \dfrac{e^{2t}-e^{-3t}}{t}$, g) $f(t) = \begin{cases} 0 & 0<t<2 \\ t-2 & 2\le t\le 3 \\ t^2/9 & t>3 \end{cases}$

Table of Inverse Laplace Transformations

F(s)	**$L^{-1}\{F(s)\} = f(t)$**	**Examples**
c/s	c	$L^{-1}\{3/s\} = 3$
$1/(s-a)$	e^{at}	$L^{-1}\{1/(s+2)\} = e^{-2t}$
$s/(s^2+a^2)$	$\cos at$	$L^{-1}\{s/(s^2+9)\} = \cos 3t$
$1/(s^2+a^2)$	$(\sin at)/a$	$L^{-1}\{1/(s^2+4)\} = (\sin 2t)/2$
$s/(s^2-a^2)$	$\cosh at$	$L^{-1}\{s/(s^2-25)\} = \cosh 5t$
$1/(s^2-a^2)$	$(\sinh at)/a$	$L^{-1}\{1/(s^2-16)\} = (\sinh 4t)/4$
$\dfrac{1}{s^{n+1}}$	$t^n/n!$	$L^{-1}\{\dfrac{1}{s^5}\} = t^4/4!$
$\dfrac{s-a}{(s-a)^2+k^2}$	$e^{at}\cos kt$	$L^{-1}\{\dfrac{s+4}{(s+4)^2+9}\} = e^{-4t}\cos 3t$

$\frac{1}{(s-a)^2+k^2}$	$(e^{at}\sin kt)/k$	$L^{-1}\{\frac{1}{(s+1)^2+3}\} = (e^{-t}\sin\sqrt{3}\,t)/\sqrt{3}$
$F(s)\,e^{-as}$	$f(t-a)U(t-a)$	$L^{-1}\{\frac{e^{-\pi s}}{s^2+9}\}=\frac{1}{3}\sin 3(t-\pi)U(t-\pi)$
$\frac{1}{(s-a)^{n+1}}$	$e^{at}\,t^n/n!$	$L^{-1}\{\frac{1}{(s+2)^4}\}= e^{-2t}\,t^3/3!$
$\frac{F(s)}{s}$	$\int_{u=0}^{t} f(u)\,du$	$L^{-1}\{\frac{1}{s(s^2+25)}\} =$ $\frac{1}{5}\int_{u=0}^{t}\sin(5u)\,du = (1-\cos 5t)/25$

Example 5: Find inverse Laplace of the following functions

a) $F(s) = \frac{s}{s^2+4s+9}$, b) $F(s) = \frac{e^{-2s}}{s+3}$, c) $F(s) = \frac{1}{(s+9)^3}$

Solution

a) $F(s) = \frac{s+2-2}{(s+2)^2+5} = \frac{s+2}{(s+2)^2+5} - \frac{2}{(s+2)^2+5}$, therefore

$$f(t) = e^{-2t}\left(\cos\sqrt{5}\,t - \frac{2}{\sqrt{5}}\sin\sqrt{5}\,t\right)$$

b) $f(t) = e^{-3(t-2)}\,U(t-2)$, c) $f(t) = \frac{1}{2!}\,t^2\,e^{-9t}$

Problems on inverse Laplace

Find inverse Laplace of the following functions

a) $F(s) = \frac{1}{(s-1)^3} - \frac{6}{s^2+9}$, b) $F(s) = \frac{4}{s^2-4s+4} + \frac{1}{(4-2s)}$

c) $F(s) = \frac{4}{s(s^2+9)} + \frac{e^{-2s}}{s^2+9}$

Laplace transform of differential equation of order n

In part one, we studied the solution of 1st order D.E. & higher order D.E. which composed of particular solution and complementary solution and in this chapter, we will study the solution of ordinary differential equations using Laplace transform. If we consider the ordinary differential equation:

$$a_0\, y^{(n)} + a_1\, y^{(n-1)} + ... + a_n\, y = f(t)$$

When initial condition at t = 0 are given can be greatly speeded up by the use of Laplace transform.

Transforms of Derivatives

a- $L\{ y`(t) \} = s\,Y(s) - y(0)$,

b- $L\{ y``(t) \} = s^2\,Y(s) - s\,y(0) - y`(0)$

In general, we have

$L\{ y^{(n)}(t) \} = s^n\,Y(s) - s^{n-1}\,y(0) - s^{n-2}\,y`(0) - \ldots - y^{(n-1)}(0)$

Proof

a) Since $L\{ y`(t) \} = \int_0^\infty e^{-st}\,y`(t)\,dt$, by using (integration by parts), so $L\{y`(t)\} = [y(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st}\,y(t)\,dt = sY(s) - y(0)$

b- Since $L\{ y``(t) \} = \int_0^\infty e^{-st}\,y``(t)\,dt$, by using (integration by parts), so $L\{ y``(t) \} = [y`(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st}\,y`(t)\,dt$

$$= s[\,[y(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st}\,y(t)\,dt] - y`(0)$$

$$= s^2Y(s) - s\,y(0) - y`(0)$$

Example 6: Solve the following differential equations:

a) $y` - 2y = 8$, $y(0) = 0$

b) $y``+ y` - 2y = 0$ , $y(0) = 0$ , $y`(0) = 3$

c) $y``- 2y` - 3y = 4$ , $y(0) = 1$ , $y`(0) = 7$

d) $y``- 3y` + 2y = 8e^{2t}$, $y(0) = y`(0) = 3$

Solution

a) Take Laplace to both sides , we will get:

L { $y` - 2y = 8$ }, hence $sY(s) - y(0) -2 Y(s) = \frac{8}{s}$, but $y(0) = 0$, therefore $(s - 2)Y(s) = \frac{8}{s} \Rightarrow Y(s) = \frac{8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$ $= \frac{A(s-2)+Bs}{s(s-2)}$. By using partial fraction so that put $s = 0$, then $A = -4$ and put $s = 2$, then $B = 4$, therefore $Y(s)= \frac{8}{s(s-2)} = \frac{-4}{s} + \frac{4}{s-2}$, hence $y(t) = 4 (e^{2t} -1)$

b) Take Laplace to both sides, we will get L { $y``+ y` - 2y = 0$ }, therefore $s^2 Y(s) - s y(0) - y`(0) + s Y(s) - y(0) - 2 Y(s) = 0$, but $y(0) = 0$, $y`(0) = 3$, thus $s^2 Y(s) - 3 + s Y(s) - 2 Y(s) = 0$, hence $(s^2 + s - 2)Y(s) = 3 \Rightarrow Y(s) = \frac{3}{(s^2 + s - 2)} =$

$\frac{3}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$ and by using partial fraction so that put s = 1, then B = 1 and Put s = -2 , then A = -1, thus Y(s) = $\frac{-1}{s+2} + \frac{1}{s-1}$, hence y(t) = ($e^{t} - e^{-2t}$).

c) Take Laplace to both sides , we will get L{y``- 2y` - 3y = 4}, therefore $s^2$ Y(s) - s y(0) - y`(0) -2[s Y(s) - y(0)] - 3 Y(s) = $\frac{4}{s}$, but y(0) = 1, y`(0) = 7, therefore s^2Y(s) – s - 7 - 2[sY(s) - 1]- 3 Y(s) = $\frac{4}{s}$.

Hence (s^2 -2s - 3) Y(s) = $\frac{4}{s}$+ s + 5 $\Rightarrow$ Y(s) = $\frac{(s^2+5s+4)}{s(s^2-2s-3)}$

$= \frac{(s+4)(s+1)}{s(s-3)(s+1)} = \frac{(s+4)}{s(s-3)} = \frac{4}{s} + \frac{B}{s-3} = \frac{(s-3)A + Bs}{s(s-3)}$, and by using partial fraction so that put s = 0, then A = $\frac{-4}{3}$, also put s = 3 , then B = $\frac{7}{3}$, thus Y(s) = $\frac{1}{3}(\frac{-4}{s} + \frac{7}{s-3})$ and therefore y(t) = $\frac{1}{3}$($7e^{3t}$ - 4).

d)Take Laplace to both sides, such that L{y``- 3y`+2y=$8e^{2t}$ }, therefore $s^2$Y(s) - sy(0) - y`(0) -3[s Y(s)-y(0)]+2Y(s) = $\frac{8}{s-2}$,

but $y(0) = y`(0) = 3$, thus $s^2\ Y(s) - 3s - 3 - 3[\ sY(s) - 3\ ] + 2\ Y(s) = \frac{8}{s-2}$, therefore $(s^2 - 3\ s + 2\ )Y(s) = \frac{8}{s-2} - 6 + 3s = \frac{8+3(s-2)^2}{s-2} = \frac{8+3(s-2)^2}{(s-2)(s^2-3s+2)} = \frac{8+3(s-2)^2}{(s-2)^2(s-1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s-1}$, therefore $(s-2)(s-1)A + (s-1)B + (s-2)^2 C = 8 + 3(s-2)^2$ so that put $s = 2$, then $B=8$, put $s =1$, then $C = 11$, and put $s = 0$, then $2A - B+4C = 20$, thus $2A-8+44 = 20$, then $A = -16$, thus $Y(s)= \frac{-16}{s-2} + \frac{8}{(s-2)^2} + \frac{11}{s-1} \Rightarrow y(t) = (11e^t - 16e^{2t} + 8t\ e^{2t}\ )$.

Solution of linear System of differential equations

It is easier with the Laplace transform method to solve simultaneous differential equations by transforming both equations and then solve the two equations in the *s*-domain and finally obtain the inverse to get the solution in the t-domain.

Example 7: Solve for x(t) and y(t), given that x(0) = 4, y(0) = 3, and $\frac{dx}{dt} + x + 4y = 10, \quad x - \frac{dy}{dt} - y = 0$

Solution

Take Laplace to the above equations such that:

s X(s) - x(0) + X(s) + 4 Y(s) = 10/s, X(s) – [s Y(s) - y(0)] - Y(s) = 0, thus (s +1) X(s) + 4 Y(s) = 10/s + 4…(1)

X(s) – (s +1)Y(s) = -3…(2)

Multiply equation (2) by (s+1) and subtract, therefore

$$[4+(s+1)^2]\ Y(s) = 10/s + 3s+7$$

Thus $Y(s) = \frac{10}{s[4+(s+1)^2]} + \frac{3(s+1)+4}{[4+(s+1)^2]} = \frac{2}{s} + \frac{(s+1)+2}{[4+(s+1)^2]}$

Hence $y(t) = 2 + e^{-t}(\cos 2t + \sin 2t)$, but $x = \frac{dy}{dt} + y$, therefore

$$x = 2+2e^{-t}\ (\cos 2t - \sin 2t)$$

Problems on Solution of linear System of D.E.

Solve the following differential equations:

1- $y``-4y` + 4y = 0$, $y(0) = y`(0) = 1$

2- $y`` + 2y` - 2y = 2t + 3$, $y(0) = 1$, $y`(0) = 1$

3- $y```+4y`` + 3y` = e^t$, $y(0) = 3$, $y`(0) = 2$, $y``(0) = 2$

4- $\frac{dx}{dt} + 5x + \frac{dy}{dt} + 3y = e^{-t}$, $\frac{dx}{dt} + 2x + \frac{dy}{dt} + y = 3$

5- $\frac{dx}{dt} + 2x + \frac{dy}{dt} + 4y = 1$, $\frac{dx}{dt} + x + \frac{dy}{dt} + 5y = 2$

References

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